

Generalized Heisenberg Algebras and Fibonacci Series

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Abstract

We have constructed a Heisenberg-type algebra generated by the Hamiltonian, the step operators and an auxiliar operator. This algebra describes quantum systems having eigenvalues of the Hamiltonian depending on the eigenvalues of the two previous levels. This happens, for example, for systems having the energy spectrum given by Fibonacci sequence. Moreover, the algebraic structure depends on two functions $f(x)$ and $g(x)$. When these two functions are linear we classify, analysing the stability of the fixed points of the functions, the possible representations for this algebra.

Keywords: Heisenberg algebra; Fibonacci series; quasi-periodic systems; Fock representation.

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1 Introduction

Due to their interesting mathematical structure and possible physical applications deformed algebras have attracted much attention in the last twenty years. In 1982 Kulish [1] showed that the underlying algebra of the XXZ -Heisenberg spin model was a deformation of the $su(2)$ algebra, called nowadays $su_q(2)$.

Following the technique developed by Schwinger to compose two Heisenberg oscillators to obtain $su(2)$ algebra, in 1989 Macfarlane and Biedenharn [2, 3] constructed the $su_q(2)$ algebra using two q -oscillators. q -oscillator algebra is a deformation of the Heisenberg algebra through a parameter q , which reproduces Heisenberg algebra when the parameter $q \rightarrow 1$.

Since Heisenberg algebra has an important role in several areas of physics it was tempting to search for applications of this new structure. Thus, there was an intense study of possible physical applications of q -oscillators and since then some successful ones have been described in the literature [4, 5, 6, 7, 8, 9, 10, 11]. They have also been used as a phenomenological approach to study composite particles [12, 13, 14, 15, 16], spectra of atoms and molecules [12, 18, 19, 6, 20, 11], generalized field theory [21, 14, 15, 16] and coherent states [22].

Recently, a new algebraic structure was proposed [24] that generalizes the Heisenberg algebra. This structure contains also q -oscillators as a particular case and has been successfully used in some different physical problems [11,

12, 14, 15, 16, 17, 22, 23]. In this new algebra, called Generalized Heisenberg Algebra (GHA), the commutation relations among the operators a , a^\dagger and J_0 depend on a general analytic functional $f(x)$ and are given by:

$$J_0 a^\dagger = a^\dagger f(J_0) \quad (1)$$

$$a J_0 = f(J_0) a \quad (2)$$

$$[a, a^\dagger] = f(J_0) - J_0, \quad (3)$$

with $a = (a^\dagger)^\dagger$ and $J_0 = J_0^\dagger$. Identifying the operator J_0 with the Hamiltonian of a physical system, this algebra tell us that the J_0 eigenvalues ($J_0|n\rangle = e_n|n\rangle$ in Fock space), which are the energy spectrum of the physical system, are obtained by an one-step recurrence ($e_n = f(e_{n-1})$), *i.e.*, each eigenvalue depends on the previous one. Thus, the eigenvalue behavior can be studied by dynamical system techniques, enormously simplifying the task of finding possible representations of the algebra [24].

When the functional $f(x)$ is linear we re-obtain the q -oscillator algebra [24, 25]; other kinds of functionals give more general algebraic structures than q -oscillators [24]. The functional $f(x)$ can be, for example, a polynomial and therefore it depends on some parameters (the polynomial coefficients). Depending on the kind of functional and their parameters, finite or infinite representations are allowed [24]. The J_0 operator eigenvalues depend on the characteristic function $f(x)$ and it can present a variety of different behaviors (monotonously increasing or decreasing, etc, upper limited or not, etc) [24]. For instance, for the functional $f(J_0) = \left(\sqrt{J_0} + \sqrt{b}\right)^2$, where $b = \pi^2/2mL^2$ with m and L being the mass and the length of the well, one obtains the

square-well potential algebra [26]. However, there are several systems which cannot be described by such a class of algebras.

Quasi-periodic structures have attracted much interest lately [27]. In particular, Fibonacci sequences were considered in several areas of mathematics and physics [27, 28]. Moreover, it was shown that an electron gas under sudden heating has a quasiequilibrium regime with a quasiperiodic energy distribution [29]. Since most systems having a quasiperiodic energy spectrum cannot be described by the GHA, in this paper we are going to present a first step in this direction. We construct here a Heisenberg-type algebra for a system having the energy spectrum described by a generalized Fibonacci sequence. In order to realize this, we propose a simple algebraic structure depending on two functionals, $f(x)$ and $g(x)$. We introduce an operator H , being the Hamiltonian of a system¹, having eigenvalues obeying the equation $e_{n+1} = f(e_n, e_{n-1})$, i.e., the eigenvalue of the n -th state depends on the two previous eigenvalues. We call this structure a *two-step algebra*. In order to extend the “one-step algebra” described by the GHA, we introduce also an additional operator J_3 .

When both functionals $f(x)$ and $g(x)$ appearing in the two-step algebra are linear functions, we show that we can re-write both the H and J_3 operators in Fock space as functions of a single number operator, N . The algebraic structure generated by these functionals contains the GHA, q -oscillator algebra and is related to other interesting algebras as special cases.

In section II we introduce the two-step generalized Heisenberg algebra. In

¹Instead of calling J_0 as it was done in Eqs. (1 - 3), we will call this operator, from now on, directly as H .

section III we study the linear case for the algebra, that contains a Fibonacci-like spectra and many other interesting quasi-periodic sequences and we also discuss the types of representations we could find. In section IV we present our conclusions and in the appendix we classify possible representations for the algebra when $f(x)$ and $g(x)$, appearing in the algebra, are linear functions.

2 Extended Two-step Heisenberg Algebra

We propose an structure generated by the operators H , J_3 , a^\dagger and a , with a and a^\dagger , $a = (a^\dagger)^\dagger$, being the step operators, $H = (H)^\dagger$ the Hamiltonian and $J_3 = (J_3)^\dagger$, obeying the following relations:

$$\left\{ \begin{array}{l} Ha^\dagger = a^\dagger(f(H) + J_3); \end{array} \right. \quad (4)$$

$$\left\{ \begin{array}{l} aH = (f(H) + J_3)a; \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} J_3a^\dagger = a^\dagger g(H); \end{array} \right. \quad (6)$$

$$\left\{ \begin{array}{l} aJ_3 = g(H)a; \end{array} \right. \quad (7)$$

$$\left\{ \begin{array}{l} [a, a^\dagger] = f(H) - H + J_3; \end{array} \right. \quad (8)$$

$$\left\{ \begin{array}{l} [H, J_3] = 0, \end{array} \right. \quad (9)$$

where we have assumed that $f(x)$ and $g(x)$ are analytical functions. In the Fock space representation of this structure, one has the normalized vacuum state $|0\rangle$ defined by the relations:

$$a|0\rangle = 0 \quad (10)$$

$$H|0\rangle = \alpha_0|0\rangle \quad (11)$$

$$J_3|0\rangle = \beta_0|0\rangle, \quad (12)$$

where α_0 and β_0 are real numbers.

A Casimir operator associated with this algebraic relations is

$$C^{(1)} = aa^\dagger - f(H) - J_3 = a^\dagger a - H, \quad (13)$$

satisfying the following relations:

$$[C^{(1)}, a^\dagger] = [C^{(1)}, a] = [C^{(1)}, H] = [C^{(1)}, J_3] = 0. \quad (14)$$

2.1 Representation Theory

In order to build a representation theory for the structure given by (4-9), let us start with the vacuum state defined by relations (10-12).

The a^\dagger operator acting on the vacuum state produces another vector, say $|k\rangle$: $a^\dagger|0\rangle = |k\rangle$. It is trivial to see that the new vector $|k\rangle$ is orthogonal to vector $|0\rangle$. Let us call $|k\rangle = N_0|1\rangle$. The N_0 constant can be determined by performing the inner product of $|k\rangle$ and its dual and by using relation (8):

$$\langle k|k\rangle = \langle 0|aa^\dagger|0\rangle = \langle 0|a^\dagger a + f(H) - H + J_3|0\rangle \Rightarrow N_0^2 = f(\alpha_0) - \alpha_0 + \beta_0. \quad (15)$$

In order to determine the H eigenvalue on the state $|1\rangle$, we have, using equation (4):

$$H|1\rangle \equiv \alpha_1|1\rangle = (f(\alpha_0) + \beta_0)|1\rangle. \quad (16)$$

Just as H , we can calculate the J_3 eigenvalue using equation (6),

$$J_3|1\rangle \equiv \beta_1|1\rangle \Rightarrow J_3 \frac{a^\dagger}{N_0}|0\rangle = \frac{a^\dagger}{N_0}g(H)|0\rangle = g(\alpha_0)|1\rangle, \quad (17)$$

therefore, we have $\beta_1 = g(\alpha_0)$.

In general, it can be proved that:

$$H|n\rangle = \alpha_n|n\rangle; \quad (18)$$

$$J_3|n\rangle = \beta_n|n\rangle, \quad (19)$$

$$a^\dagger|n\rangle = N_n|n+1\rangle; \quad (20)$$

$$a|n\rangle = N_{n-1}|n-1\rangle; \quad (21)$$

where

$$\alpha_{n+1} = f(\alpha_n) + \beta_n, \quad (22)$$

$$\beta_{n+1} = g(\alpha_n), \quad (23)$$

$$\begin{aligned} N_{n+1}^2 &= N_n^2 + f(\alpha_{n+1}) - \alpha_{n+1} + \beta_{n+1} \\ &= \alpha_{n+2} - \alpha_0. \end{aligned} \quad (24)$$

If we combine equations (22) and (23), the two-step dependence of the H eigenvalue becomes clear: $\alpha_{n+1} = f(\alpha_n) + g(\alpha_{n-1})$, where $\alpha_0 \leq \alpha_n$, for all $n > 0$. Choosing different functions and different initial values α_0 and β_0 , we can construct different Fock space representations for this mathematical structure. In matrix representation, the operators of the algebra can be written as

$$H = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & \alpha_2 & 0 & \dots \\ 0 & 0 & 0 & \alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad J_3 = \begin{pmatrix} \beta_0 & 0 & 0 & 0 & \dots \\ 0 & \beta_1 & 0 & 0 & \dots \\ 0 & 0 & \beta_2 & 0 & \dots \\ 0 & 0 & 0 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ N_0 & 0 & 0 & 0 & \dots \\ 0 & N_1 & 0 & 0 & \dots \\ 0 & 0 & N_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a = (a^\dagger)^\dagger. \quad (25)$$

In the next section, we shall study the case where $f(x)$ and $g(x)$ are linear functions.

3 The linear case - Generalized Fibonacci Algebra

If we assume $f(x) = rx$ and $g(x) = sx$ ($r, s \in \mathbb{R}$), the equations (4-9) reduce to

$$Ha^\dagger = a^\dagger(rH + J_3) \quad (26)$$

$$aH = (rH + J_3)a \quad (27)$$

$$J_3a^\dagger = sa^\dagger H \quad (28)$$

$$aJ_3 = sHa \quad (29)$$

$$[a, a^\dagger] = (r - 1)H + J_3 \quad (30)$$

Using Eqs. (22) and (23) for this linear case we get:

$$\beta_{n+1} = s\alpha_n. \quad (31)$$

$$\alpha_{n+1} = r\alpha_n + \beta_n = r\alpha_n + s\alpha_{n-1}. \quad (32)$$

The recurrence equation (32) is a difference equation which generates a Generalized Fibonacci Sequence. Hence, the sequence of eigenvalues of the H

operator follows a Generalized Fibonacci Series. We can write equations (31) and (32) as a two-dimensional linear map, which is more adequate in order to analyze the representation theory of the algebra. This map can be written as:

$$\begin{pmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}. \quad (33)$$

In [24], the authors have analyzed, for one quantum number, the representation theory of the GHA using dynamical systems techniques. The representations were obtained analyzing the stability of the fixed-points and the cycles of the equation $\alpha_{n+1} = f(\alpha_n)$. In this work we have extended the previous effort and analyzed the stability of the fixed-points of the two dimensional system given by Eq. (33).

3.1 Stability Analysis and Representation Theory

The fixed-points of equation (33) can be found by solving the system

$$\begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}. \quad (34)$$

The solutions of the previous equation are $(\alpha^*, \beta^*) = (0, 0) \forall r, s \in \mathbb{R}$ and $(\alpha^*, \beta^*) = (\alpha^*, s\alpha^*)$ for $r + s = 1$, where $r, s, \alpha^* \in \mathbb{R}$. The stability of these fixed points are given by the eigenvalues of the 2×2 -matrix in Eq. (34), which can be calculated by means of the characteristic equation, $\lambda^2 - r\lambda - s = 0$. These eigenvalues are independent of a specific point in the (α, β) -space, as the recurrence given by Eq. (33) is linear, and can be written as:

$$\lambda_{\pm} = \frac{r \pm \sqrt{r^2 + 4s}}{2}. \quad (35)$$

The main possibilities are:

- (i) the fixed point (α^*, β^*) is asymptotically stable if $|\lambda_{\pm}| < 1$; an initial point (α_0, β_0) , iterated using Eq. (33), approaches (α^*, β^*) as the number of iterations increases;
- (ii) if at least one of the eigenvalues is, in modulus, bigger than one, the fixed point (α^*, β^*) is called unstable and the iterations of an initial point (α_0, β_0) by Eq. (33) move this point far away the fixed point.
- (iii) if both eigenvalues are, in modulus, equal to one - edge BC in Figure (3) - we call the fixed point $(\alpha^*, \beta^*) = (0, 0)$ marginally stable . The iterations of an initial point in the (α, β) -space will not move it far away the fixed point, but it will not approach the fixed point too as the iterations increase.

In Figure (3) we show the stability regions of the fixed points $(\alpha^*, \beta^*) = (0, 0)$ and $(\alpha^*, \beta^*) = (\alpha^*, s\alpha^*)$ in the parameter space (r, s) .

When $r + s \neq 1$ the fixed point $(\alpha^*, \beta^*) = (0, 0)$ is the only fixed point in the (α, β) space. It is asymptotically stable for values of r and s inside the triangle ABC and unstable for values of r and s outside this triangle. It is marginally stable on the edge BC and, on the edge AB , where one eigenvalue (λ_-) is always equal to -1 and the other (λ_+) is equal to $1 + r$, with $r \in [-2, 0]$, the iteration of any initial point (α_0, β_0) will approach a cycle two, with the line joining them crossing the fixed point $(0, 0)$ in the (α, β) -space.

If $r + s = 1$, the dotted line in Figure (3), there are a line of fixed points in the (α, β) -space, $(\alpha^*, \beta^*) = (\alpha^*, (1-r)\alpha^*)$, with $\alpha^* \in \mathbb{R}$. All these fixed points are unstable if, in the line $r + s = 1$, $r < 0$ or $r > 2$, i.e., for points in this line outside the edge AC (see Figure (3)). For points (r, s) in this line belonging

to the edge AC , the associated eigenvalues are $\lambda_+ = +1$ and $\lambda_- = r - 1$, with $r \in [0, 2]$. The fixed points in the (α, β) -space, $(\alpha^*, \beta^*) = (\alpha^*, (1 - r)\alpha^*)$, are stable in one direction, crossing the line of fixed points, associated with the eigenvalue $\lambda_- = r - 1$, $r \in (0, 2)$. The direction along the line of fixed points is marginally stable, associated with the eigenvalue $\lambda_+ = +1$.

In the linear case, given the values of r and s (or, equivalently, the values of λ_+ and λ_- , see Fig. 5), the possible values of α_0 and β_0 for the representations of the linear case of the algebra we are analyzing are presented in the appendix. If the two-dimensional map, Eq. (33), has an unstable fixed point (i.e., if the values of r and s are outside the triangle of Fig. 3), the possible representations are infinite-dimensional and their corresponding spectra have no upper limit; the difference between two consecutive numbers in the corresponding sequence increases as the number of iterations increases. If this map has an asymptotically stable fixed point (i.e., if the values of r and s are inside the triangle of Fig. 3), the possible representations are also infinite-dimensional but they have superiorly-limited spectra; the difference between two consecutive numbers in the corresponding sequence decreases as the number of iterations increases. The representations associated with marginally stable cases are more complex and, besides infinite-dimensional representations, it is also possible to have finite-dimensional ones. See the appendix for a discussion of the possible values of α_0 and β_0 for these representations.

3.2 Fibonacci series

If we assume $r = s = 1$ the relation given by equation (32) becomes

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1}, \quad (36)$$

which yields the usual Fibonacci sequence if we choose $(\alpha_0, \beta_0) = (1, 0)$. The Fibonacci series have as eigenvalues $\lambda_{\pm} = (1 \pm \sqrt{5})/2$. The $\lambda_+ (> 1)$ eigenvalue is the so-called golden-number. The single fixed-point $(\alpha^*, \beta^*) = (0, 0)$ is, therefore, unstable. The representations are infinite dimensional and the sequences (depending on the initial values (α_0, β_0)) increase without an upper limit, as we know. Specific values of α_0 and β_0 describing possible representations for this case will be discussed in the next section. The values of α_n and β_n for $r = s = 1$ are shown in Table 1. The eigenvalues α_n and β_n provide the number of elements of kind “A” and “B”, respectively, in the n -th inflation of the Fibonacci chain (Figure 1). Thus, the operator a^\dagger acts as an “inflation operator” of the Fibonacci chain, while the H eigenvalues provide the number of elements of the quasiperiodic chain generated at the n -th inflation.

4 Fock space two-step Heisenberg algebra

The Generalized Fibonacci Series (Eq. (32)) can be written by means of the so-called Binet formula:

$$\alpha_n = \alpha_0 \frac{p^{n+1} - q^{n+1}}{p - q} + \beta_0 \frac{p^n - q^n}{p - q} \quad (37)$$

$$\equiv \alpha_0 [n + 1]_{p,q} + \beta_0 [n]_{p,q}, \quad (38)$$

where $(p, q) \equiv (\lambda_+, \lambda_-)$ are the roots given by Eq. (35), $n \geq 1$ and $[n]_{p,q} = \frac{p^n - q^n}{p - q}$ is the two-parameter p, q -number (Gauss Number) [35]. As H is the Hamiltonian, the choice of β_0 is not completely free inasmuch as we should guarantee that $\alpha_n \geq \alpha_0$ for any $n > 0$ (the energy of the ground state must be the lowest one). This condition implies that, once the value of α_0 had been chosen, the allowed values of β_0 should satisfy the relation

$$\beta_0 [n]_{p,q} \geq (1 - [n + 1]_{p,q}) \alpha_0, \quad (39)$$

for any value of n , $n > 0$. In particular, if $[n]_{p,q} > 0$ for any value of $n > 0$, the condition to be satisfied by β_0 can be written as:

$$\beta_0 \geq \frac{1 - [n + 1]_{p,q}}{[n]_{p,q}} \alpha_0, \quad (40)$$

for any value of n , $n > 0$.

The solution for β_0 for the whole parameter space will be described in the appendix. For the specific case of Fibonacci chain ($r = s = 1$) it is simple to see that the term $(1 - [n + 1]_{p,q})/[n]_{p,q}$ varies from $-(1 + \sqrt{5})/2$ up to 0. Then, $\alpha_0 \geq 0$ implies $\beta_0 \geq 0$ and $\alpha_0 < 0$ implies that $\beta_0 \geq ((1 + \sqrt{5})/2) |\alpha_0|$.

We can show that there exist another Casimir Operator $C^{(2)}$ when $f(x)$ and $g(x)$ are linear functions:

$$C^{(2)} = aa^\dagger - \alpha_0([N + 2]_{p,q} - 1) - \beta_0([N + 1]_{p,q} - 1), \quad (41)$$

where $[N]_{p,q} = \frac{p^N - q^N}{p - q}$ is the p, q -deformed number operator and N is the usual number operator, defined in Fock space as

$$N|n\rangle = n|n\rangle. \quad (42)$$

Using Eq. (37), it is possible to reduce the algebraic structure (26-30) to a Heisenberg-like structure composed by the usual operators A , A^\dagger and N . From Eq. (37) H can be written in Fock space as:

$$H \rightarrow \alpha_0 \frac{p^{N+1} - q^{N+1}}{p - q} + \beta_0 \frac{p^N - q^N}{p - q}, \quad (43)$$

and, using Eq. (31), J_3 can be written as

$$J_3 \rightarrow s \left(\alpha_0 \frac{p^N - q^N}{p - q} + \beta_0 \frac{p^{N-1} - q^{N-1}}{p - q} \right). \quad (44)$$

Note that the effect of the application of H and J_3 on $|n\rangle$ is exactly the same as if we use Eqs. (43) and (44) on $|n\rangle$. Substituting H and J_3 in Eq. (30) by Eqs. (43) and (44) and using Eqs. (20), (21) and (42) we get:

$$[N, a^\dagger] = a^\dagger, \quad (45)$$

$$[N, a] = -a, \quad (46)$$

$$\begin{aligned} [a, a^\dagger] &= (r - 1) (\alpha_0 [N + 1]_{p,q} + \beta_0 [N]_{p,q}) + \\ &+ s (\alpha_0 [N]_{p,q} + \beta_0 [N - 1]_{p,q}). \end{aligned} \quad (47)$$

The above algebraic structure is the Fock space form of a two-step Heisenberg algebra. Notice that the algebraic structure described in Eqs. (26-30) correspond to Eqs. (45-47) plus the Hamiltonian $H = \alpha_0 [N + 1]_{p,q} + \beta_0 [N]_{p,q}$, whose eigenvalues follow the generalized Fibonacci sequence.

Remembering that $p, q = -s$ and $p + q = r$ (p and q are roots of $x^2 - rx - s = 0$), the right hand side of Eq. (47) can be written as

$$(p + q - 1)(\alpha_0 [N + 1]_{p,q} + \beta_0 [N]_{p,q}) - pq(\alpha_0 [N]_{p,q} + \beta_0 [N - 1]_{p,q}), \quad (48)$$

which, after some algebra allows us to write Eq. (47) as:

$$[a, a^\dagger] = \alpha_0 ([N + 2]_{p,q} - [N + 1]_{p,q}) + \beta_0 ([N + 1]_{p,q} - [N]_{p,q}). \quad (49)$$

Considering $\alpha_0 = 1$ and $\beta_0 = 0$ in Eq. (49), Eqs. (45-46) and (49) can be related to the (p, q) -oscillator, reference [35], which was used in nuclear spectroscopy [36]. Thus, the H eigenvalues of the (p, q) -oscillator can be written as a particular case of the generalized Fibonacci numbers with “initial conditions” $\alpha_0 = 1$ and $\beta_0 = 0$. In our “extended algebra”, the H eigenvalues are written as Generalized Fibonacci numbers where the initial values α_0 and β_0 can assume any value since that $\alpha_0 \leq \alpha_n$ for any $n > 0$. Thereby, each set of initial values (α_0, β_0) define a new representation.

5 Final comments

We have constructed a Heisenberg-type algebra having as generators the Hamiltonian, the step-operators and an auxiliary operator. Moreover, this structure depends on two analytic functions $f(x)$ and $g(x)$. Since the energy spectra of the systems described by this algebra depend on the two previous energy levels, this algebra was called generalized two-step Heisenberg algebra.

This algebra describe quantum systems having eigenvalues of the operator H (Hamiltonian) analytically depending on the two previous eigenvalues, i. e., $\epsilon_{i+1} = h(\epsilon_i, \epsilon_{i-1})$, where $h : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ is a function and ϵ_i is the eigenvalue of H acting on the eigenstate i , i. e., $H|i\rangle = \epsilon_i|i\rangle$. A quantum system having the Hamiltonian eigenvalues obeying the Fibonacci sequence (quasi-periodic systems) could, for example, be described by this algebra. Possible spectra generated by this algebra can be seen in Figure 4, showing that the typical possible spectra found in nature can be generated by this algebraic structure.

We have discussed the representations of this algebra and we have classi-

fied in detail the possible representations for linear $f(x)$ and $g(x)$. To perform this classification we have used tools from dynamical system techniques as the analysis of the stability of the fixed points of the characteristic functions of the algebra, $f(x)$ and $g(x)$.

In [12], an algebraic phenomenological approach, based on the GHA, was implemented for the CO_2 molecule. We think that the algebra constructed here could be used to generalize the mentioned approach to more general spectra.

We also consider mathematically interesting to analyse in detail the representations of this algebra at least for simple nonlinear examples of the characteristic functions.

Appendix

In this appendix we discuss the representations associated with the linear case. In order to classify the possible values of α_0 and β_0 according to the value of the eigenvalues λ_+ (or p) and λ_- (or q) we divide the plane (λ_-, λ_+) in regions labeled by the numbers I, II, III, IV, V and VI , see figure 5. As, by definition, $\lambda_+ \geq \lambda_-$, only the semi-plane equal and above the diagonal line at 45° in this plane makes sense. The representations of the extended algebra can be obtained providing α_0 and β_0 satisfy the conditions given by Eqs. (39) and (40). Analysing these equations for the regions labeled from I to VII in Figure 5, it is possible to obtain representations for the extended algebra since α_0 and β_0 satisfy the following conditions (the results for regions IV, VI and VII are numerical):

Region I: $\lambda_+ > 1$ and $-1 \leq \lambda_- \leq \lambda_+$

a) if $\alpha_0 \geq 0 \rightarrow \beta_0 \geq (1 - \lambda_+ - \lambda_-)\alpha_0$

b) if $\alpha_0 < 0 \rightarrow \beta_0 \geq \lambda_+|\alpha_0|$

Region II: $0 < \lambda_+ < 1$ and $-\lambda_+ < \lambda_- < \lambda_+$

$\alpha_0 \leq 0 \rightarrow \beta_0 \geq (\lambda_+ + \lambda_- - 1)|\alpha_0|$

Region III: $-1 < \lambda_+ < 1$ and $-1 < \lambda_- < \min(-\lambda_+, \lambda_+)$

$\alpha_0 = 0$ and $\beta_0 \geq 0$

Region IV: $\lambda_+ > 1$ and $-\lambda_+ < \lambda_- < -1$

a) if $\alpha_0 \geq 0 \rightarrow \beta_0 \geq (1 - \lambda_+ + |\lambda_-|)\alpha_0$

b) if $\alpha_0 < 0 \rightarrow \beta_0 \geq (\frac{\lambda_+^2 + |\lambda_-|^2 - \lambda_+|\lambda_-| - 1}{\lambda_+ - |\lambda_-|})|\alpha_0|$

Region V: $0 < \lambda_+ < -\lambda_-$ and $\lambda_- < \min(-\lambda_+, -1)$

In this region there is no simple general analytical solution for the values of α_0 and β_0 and the possible representations should be computed case by case.

Region VI: $\lambda_- < \lambda_+ < 0$ and $\lambda_- < \min(-1, \lambda_+)$

a) if $\alpha_0 \geq 0 \rightarrow \beta_0 \geq (1 + |\lambda_+| + |\lambda_-|)\alpha_0$

b) if $\alpha_0 < 0 \rightarrow \beta_0 \geq (\frac{\lambda_+^2 + \lambda_-^2 - |\lambda_+||\lambda_-| - 1}{|\lambda_-| - |\lambda_+|})|\alpha_0|$

Region VII: $\lambda_+ < -1$, $\lambda_- < -1$ and $|\lambda_-| > |\lambda_+|$.

a) if $\alpha_0 \geq 0 \rightarrow \beta_0 \geq (1 + |\lambda_+| + |\lambda_-|)\alpha_0$

b) if $\alpha_0 < 0 \rightarrow \beta_0 \geq -|\lambda_-||\alpha_0|$

The possible expressions for α_0 and β_0 belonging to the boundaries between these regions are the following:

i) Boundary between region *I* and *II*:

$\beta_0 \geq -\lambda_- \alpha_0$ for $\alpha_0 \in \Re$

ii) Boundary between region *I* and *IV*:

The same expressions for α_0 and β_0 allowed for region *I* with $\lambda_- = -1$.

iii) Boundary between region *II* and *III*:

The same expressions for α_0 and β_0 allowed for region *III* with $\lambda_- = -\lambda_+$.

iv) Boundary between region *IV* and *V*:

$$\beta_0 \geq \alpha_0.$$

v) Boundary between region *V* and *VI*:

The same expressions for α_0 and β_0 allowed for region *VI* with $\lambda_+ = 0$.

vi) Boundary between region *III* and *V*:

a) If $\alpha_0 \geq 0 \rightarrow \beta_0 \geq (2 + \lambda_+) \alpha_0$

b) If $\alpha_0 < 0 \rightarrow \beta_0 \geq \lambda_+ |\alpha_0|$.

vii) Boundary between region *III* and *VI*:

The same expressions for α_0 and β_0 allowed for region *VI* with $\lambda_- = -1$.

viii) Boundary between region *VI* and *VII*:

The same expressions for α_0 and β_0 allowed for region *VI* with $\lambda_+ = -1$.

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$$A \rightarrow AB \rightarrow ABA \rightarrow ABAABA \rightarrow ABAABAABA \rightarrow \dots$$

Figure 1: The Fibonacci Chain. The substitution rule for build this chain is: $A \rightarrow AB$ and $B \rightarrow A$.

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n	$\beta_n = \alpha_{n-1}$	$\alpha_n = \alpha_{n-1} + \beta_{n-1}$
0	β_0	α_0
1	α_0	$\alpha_0 + \beta_0$
2	$\alpha_0 + \beta_0$	$2\alpha_0 + \beta_0$
3	$2\alpha_0 + \beta_0$	$3\alpha_0 + 2\beta_0$
4	$3\alpha_0 + 2\beta_0$	$5\alpha_0 + 3\beta_0$

Table 1: Eigenvalues of H and J_3 operators when $r = s = 1$.

$$A \rightarrow ABA \rightarrow ABAAABA \rightarrow ABAAABAABAABAABA \rightarrow \dots$$

Figure 2: A Fibonacci-like chain. The substitution rule for build this chain is: $A \rightarrow ABA$ and $B \rightarrow A$.

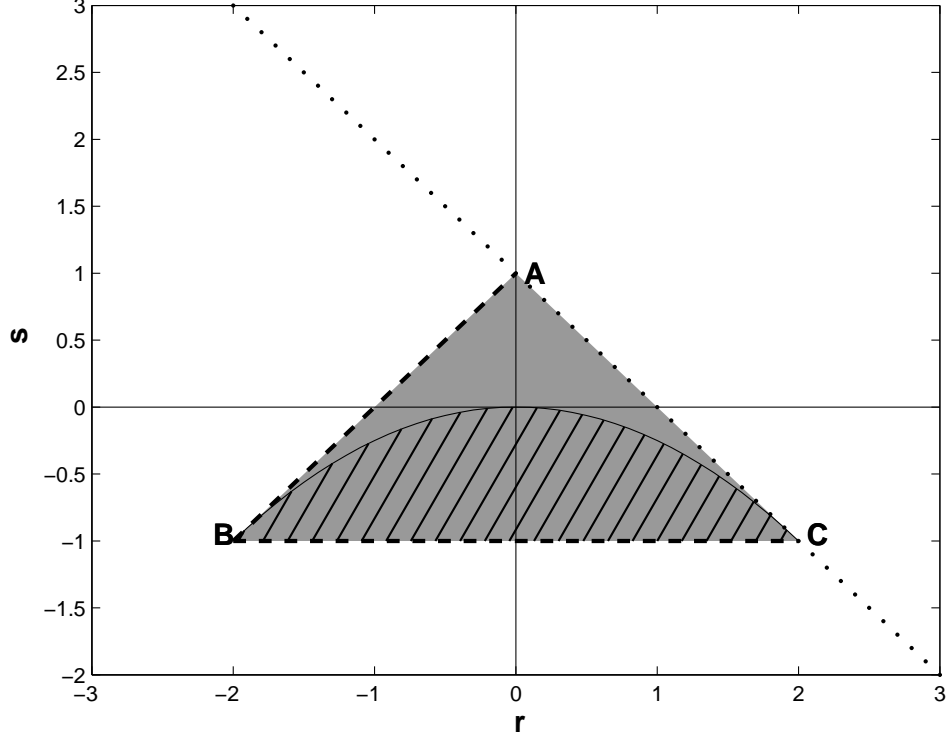


Figure 3: Stability regions in the parameter (r, s) -space. The asymptotically stable region of the $(\alpha = 0, \beta = 0)$ fixed-point, i.e., $|\lambda_{\pm}| < 1$, lies inside the triangle ABC (edges are not included). Outside the triangle ABC the $(0, 0)$ fixed-point in the (α, β) -space is unstable (at least one eigenvalue has modulus greater than one). The hatched area corresponds to the asymptotically stable region where the eigenvalues are complex. The marginally stable region of the $(0, 0)$ fixed-point lies on the dashed edges. On the edge BC both eigenvalues λ_{\pm} are complex, with unitary modulus. The dotted line is the region in the parameter space allowing the existence of the fixed-points $(\alpha^*, s\alpha^*)$ in the dynamical space (α, β) . These fixed points are marginally stable for values of the parameters (r_2, s) on the edge AC and unstable outside. The eigenvalues at the points A , B , and C are, respectively, $(1, -1)$, $(-1, -1)$ and $(1, 1)$.

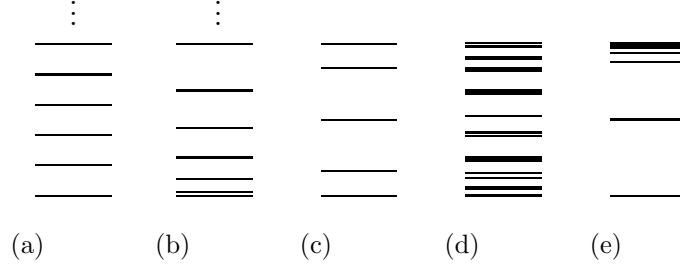


Figure 4: Examples of energy levels for various parameters of the algebra. In (a), for $r = 2$ and $s = -1$, the levels are evenly spaced. In (b), for $r = 3$ and $s = -2$ the level spacing is increasing. In (c), the levels are periodic for $r = 2 \cos \tau$ and $s = -1$, where $\tau = 2\pi/k$, with $k = 3, 4, \dots$. In (d), the level spacing form a dense set for $r = 2 \cos \tau$ and $s = -1$, where $\tau = 2\pi\gamma$, with γ being an irrational number. In (e), $r = 3/2$ and $s = -1/2$, the level spacing is decreasing.

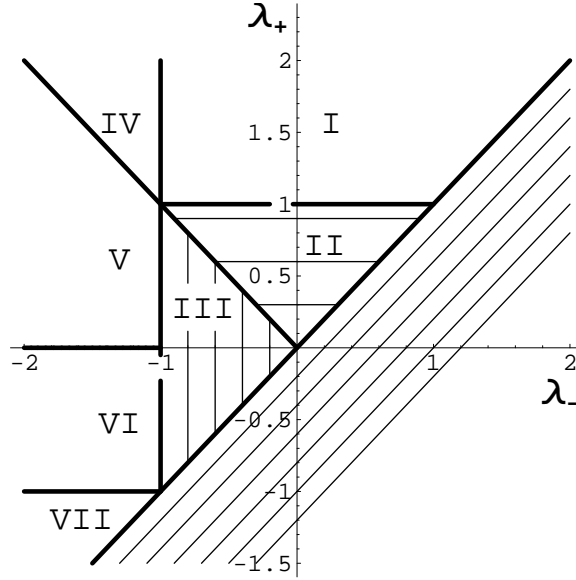


Figure 5: Regions in the (λ_-, λ_+) -plane associated with different conditions on (α_0, β_0) allowing lowest vacuum eigenstate representations.